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A CELLULAR ANALOGY FOR THE ELASTIC–PLASTIC SAINT-VENANT TORSION PROBLEM

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Abstract—The Saint-Venant torsion problem is solved by a "cellular analogy" in which solid sections are idealized as multi-celled structures. The shear flows in the cells are determined by the enforcement of geometric compatibility on the complete set of cells. The method thus provides a means for the direct determination of shear flows and stresses and is shown to provide comparable accuracy to a warping displacement approach in the case of elastic analysis. The method is also shown to be particularly convenient for elastic–plastic analysis. For a pure elastic–pure plastic material, the shear flow becomes constant once the shear stress in a cell has reached its yield value. The analysis therefore simplifies as plasticity progresses, which permits the efficient determination of the elastic–plastic response. A rectangular section example is used to demonstrate the ability of the method to accurately predict the elastic–plastic response of rectilinearly bounded sections without internal corners. Sections with internal corners are modelled less accurately due to the presence of maximal elastic shear stresses which are not parallel to the sides of the section. This feature is exemplified by the consideration of an I-section.

INTRODUCTION

For geometrically regular sections, the elastic Saint-Venant torsion problem may be solved analytically using either a stress or a displacement function approach[1, 2]. Numerical methods, for example the finite element technique, may also be applied on either basis[3, 4] for the analysis of irregular sections. Warping displacement formulations are generally more convenient for the treatment of hollow sections and are more readily extended to deal with problems involving non-uniform torsion[5], although hybrid approaches[6] are sometimes required to improve the satisfaction of the zero normal shear stress boundary condition.

In Saint-Venant (unrestrained) torsion analysis, the shear stress distribution is generally the feature of primary interest. Regardless of whether a stress or a warping function is being employed, shear stress evaluation requires a subsidiary calculation following the determination of the selected function. Feldman[7] has proposed an alternative approach whereby the shear stress components are used as the primary variables of the problem and are thereby determined directly. The method suggested by Feldman is not readily adapted to an automated analysis since it requires the formation of a set of combined equilibrium and compatibility equations in a manner which is difficult to systematize. Also, the resulting equations are neither symmetric nor banded so that the achievable computational efficiency is low.

In this paper, Feldman's direct shear stress determination approach is given a physical interpretation as a "cellular analogy". This interpretation immediately suggests the use of the shear flow theory as employed in multi-celled, thin-walled section analysis. Such an analysis[8], assumes constant shear flows in each cell so that longitudinal equilibrium is automatically satisfied and the shear flows may be determined from the enforcement of the compatibility requirements of the individual cells. The method is therefore, in essence, a flexibility method and the principal matrix is a flexibility one which is both symmetric and banded. Furthermore, as demonstrated here, the formulation is readily extended to the determination of the rotational (torsional constant) response and to elastic–plastic analysis.

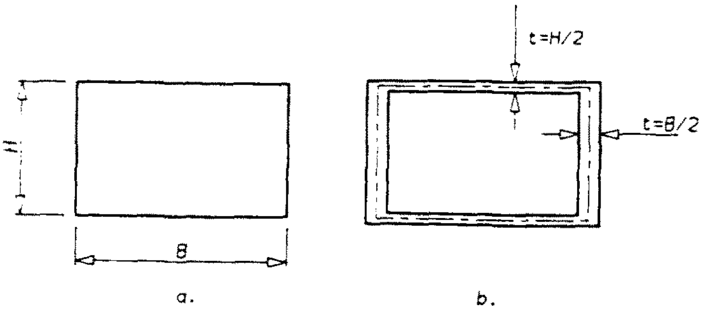

Fig. 1. Rectangular cell analogy.

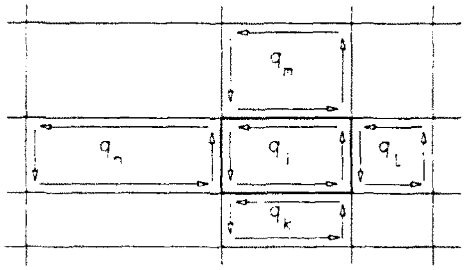

Fig. 2. Typical shear flows in adjacent cells.

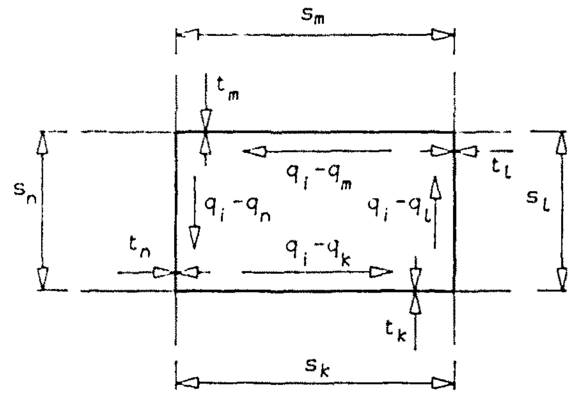

Fig. 3. Resultant shear flows in a typical cell.

ELASTIC ANALYSIS

The cellular analogy

It is presumed that the stress analysis of a prismatic, homogeneous rod subjected to uniform torsion is required on the assumption that the displacement magnitudes and material properties are such that a linear response will be obtained. The rectangular portion of the section shown in Fig. 1(a) will be idealized as the equivalent rectangular cell shown in Fig. 1(b), where the structural material has been concentrated into equivalent thin walls at the boundaries of the cell. The complete section is sub-divided, in this way, into n cells, and constant shear flows are presumed to exist in each cell. A typical internal cell, i, will be adjacent to four further cells as shown in Fig. 2 and the resultant shear flows in the walls of cell i will be as shown in Fig. 3, where the thicknesses of the cell walls have been determined by application of the analogy given in Fig. 1. The resultant shear flows of Fig. 3 will satisfy the equilibrium requirements in the longitudinal direction of the rod and the compatibility of the cell will be ensured if

$$\oint \left(\frac{q}{t}\right) \mathrm{d}s = 2A_i G\theta' \tag{1}$$

where q is the resultant shear flow, t the cell wall thickness, s is measured along the cell walls, A_i the area enclosed by cell i, G the modulus of rigidity, and θ' the uniform rate of twist.

By applying eqn (1) to each cell in turn, a complete set of compatibility equations may be established in the form

$$\sum_{j=k,l,m,n} \left(\frac{s_j}{t_j}\right) q_i - \sum_{j=k,l,m,n} \left(\frac{s_j}{t_j}\right) q_j = 2A_i G\theta' \quad (i = 1\text{–}n) \tag{2}$$

or

$$\mathbf{F}\mathbf{q} = \mathbf{a} \tag{3}$$

where

$$\mathbf{q}^t = \{q_1, \ldots, q_i, \ldots, q_n\}$$
$$\mathbf{a}^t = 2G\theta'\{A_1, \ldots, A_i, \ldots, A_n\}.$$

The solution of eqn (3) for the unknown shear flows, $\mathbf{q}$, enables the shear stress response to be obtained directly. The resultant shear flow in a given cell wall is divided by the wall's thickness to obtain the shear stress component, τ_s, in the direction of the shear flow, since, by the definition of shear flow

$$q = \tau_s t. \tag{4}$$

The rotational response may also be readily determined since, for the overall equilibrium of the section, the applied torque, T, is given by[8]

$$T = 2\sum_{i=1}^{n} A_i Q_i \tag{5}$$

so that the torsion constant, J, may be obtained as

$$J = \frac{T}{G\theta'} = \frac{2\sum_{i=1}^{n} A_i q_i}{G\theta'}. \tag{6}$$

Numerical examples

The cellular analogy method has been applied to the elastic torsional analysis of a square section, using, on account of symmetry, an eighth of the section, in which the zero resultant shear flows along the lines of symmetry were modelled by the use of appropriate zero s/t ratios in eqn (3). To compare the performance of the cellular analogy with other approaches, the square section was re-analysed with progressively finer sub-divisions by both the cellular analogy and by the finite element method using linear triangular elements based on both warping and stress functions. The results of these analyses are given in Table 1 from which it may be seen that, for this section, the performance of the cellular analogy is slightly inferior to that of the finite element method based on a warping function and that both of these solutions are superior to the stress function based finite element approach.

The evaluation of the cellular technique for more demanding elastic torsional problems was investigated by a consideration of the L-section shown in Fig. 4(a). Symmetry was again employed to reduce the analysis to one half of the section, and comparative solutions were obtained by both stress and displacement function finite element formulations, applied

Table 1. Square ($a \times a$) section comparison

	τ_{max} ($\times T/a^3$) $\tau_{max} = 4.80T/a^3$ (exact[2])		
No. of variables	Cell analogy	Warping function	Stress function
6	4.22	4.61	3.88
15	4.58	4.72	4.20
28	4.69	4.76	4.35
55	4.75	4.78	4.48

to both linear triangular elements and quadratic isoparametric elements. All the linear element analyses were based on a similar, regular mesh, whilst the mesh for the quadratic analyses was as shown in Fig. 4(b).

Table 2 compares results from the analyses with a reference solution obtained from a quadratic finite element solution, based on a finer net, for which the stress and displacement formulations gave similar results. It may be observed that the performance of the cellular analogy is again comparable to that of the linear warping displacement finite element approach. The linear stress function analysis is again inferior but may be seen to become competitive if use is made of a quadratic formulation, which significantly improves the modelling of the stress function.

It may therefore be concluded that the cellular analogy will provide a stress solution which is comparable to that obtainable from a linear warping function based approach. The advantage of using the analogy is therefore not in its improved accuracy but in the simplicity of the formulation which requires minimal effort in the establishment of the required linear equations, eqns (3), and in the post-equation solution stress determination. The use of stress variables is also advantageous in elastic–plastic analysis as will be illustrated in the next section.

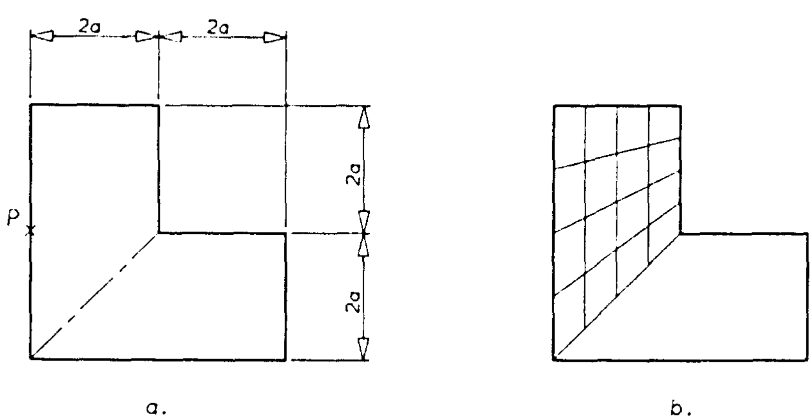

Fig. 4. L-Section analysis.

Table 2. L-Section comparison

Method	No. of variables	τ at P (Fig. 4(a)) ($\times T/a^3$)
Cellular analogy	100	0.142
Linear FE—warping function	108	0.138
Linear FE—stress function	84	0.133
Quadratic FE—warping function	56	0.150
Quadratic FE—stress function	40	0.149
Reference FE	—	0.147

ELASTIC-PLASTIC ANALYSIS

The analysis of the previous section will be extended using the previous structural model with modified material behaviour characteristics.

(a) The material is perfectly elastic–plastic so that: (i) for any shear strain $\gamma < \gamma_p$: $\tau = G\gamma$; (ii) for any shear strain $\gamma \geqslant \gamma_p$: $\tau = \tau_p$; where subscript p indicates pure plastic.

(b) Elastic unloading (known to occur in some situations[9]) does not take place.

(c) The directions of *all* the plastic shear stresses and of the *maximum* elastic stress component are parallel to a cell wall direction.

(d) Plasticity extends either from a section boundary or from an existing plastic zone.

The cellular analogy allows a particularly simple elastic–plastic formulation to be established on the basis of these assumptions. In an elastic zone, the equilibrium and compatibility requirements of eqn (3) will be fulfilled in conjunction with the material law given by assumption (a)(i). In a plastic zone, equilibrium and the material law of assumption (a)(ii) will be fulfilled by the incorporation in yielded cells of constant shear flows, which are just adequate to maintain the pure plastic shear stress in the critical walls of the cells. Since the analogy does not involve deformations in plastic zones, a flow rule is not explicitly invoked and the solutions obtained will be lower bounds, the accuracy of which will depend on the closeness to the "exact" stress pattern of the stress distributions permitted by assumptions (c) and (d).

The computational procedure adopted is to first undertake an elastic analysis at an arbitrary rotation parameter. From this analysis, it follows, by assumption (c), that an examination of the shear stresses in the walls of the cells will identify a critical cell in which the maximum elastic shear stress is present. Assumptions (a) and (c) then permit a linear extrapolation to be used to establish directly the precise rotation at which the shear stress in the critical cell just reaches the pure plastic value. It then follows from assumptions (a), (b) and (d) that the shear flow in the critical cell will remain constant under increasing rotation in order that the shear stress in the critical wall shall remain at the pure plastic value. In all subsequent analyses, the number of unknown shear flows is therefore reduced by one and substitution of its constant value is made for the shear flow in the critical cell. With this amendment, the analysis is therefore repeated at a specified, slightly increased rotation, from the results of which a further critical cell may be identified. This new critical cell is treated in the same way as the previous one, and repetition of the sequence allows the complete elastic-plastic response to be followed, as illustrated in the examples which follow.

Rectangular section

The elastic–plastic behaviour of a rectangular section of proportions 2:1 has been examined. A quarter of the section was analysed, using 50 square cells in one case and 200 square cells in a second analysis. The response of the section to uniform torsion is illustrated in Fig. 5, where the results of the cellular analogy analyses are compared with an existing analytical solution[10]. The results are presented nondimensionally, having been scaled in proportion to the fully plastic (T_p) torque predicted by the sand-heap[11] analogy.

Both cell sub-divisions may be seen to produce similar predictions which are in close agreement with the analytically calculated response. The development of plasticity in the section (based on the 200 cell analysis) is illustrated in Fig. 6. Plasticity may be seen to originate from the location of the maximum elastic shear stress in the middle of the longer side and to progressively develop towards the fully plastic distribution predicted by the sand-heap analogy.

I-*Section*

A quarter of an I-section was analysed, first using 196 similar square cells, as shown in Fig. 7(a), and second using an irregular net of 371 cells (Fig. 7(b)). The results of the two analyses are shown in Fig. 8, together with results from a published finite element analysis[5]. The initiation of plasticity, indicated by the lowest designated points in each

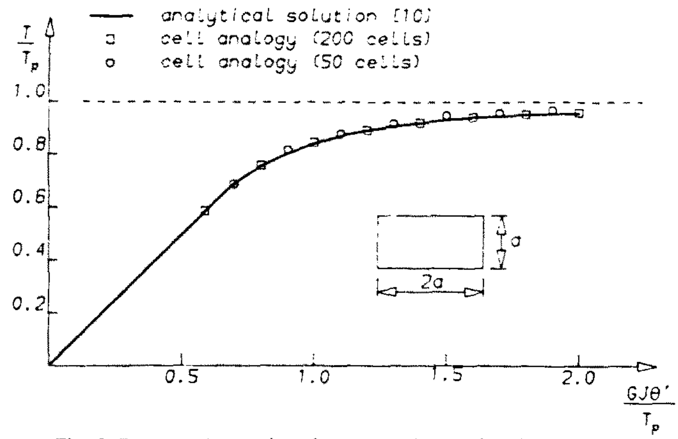

Fig. 5. Rectangular section elastic plastic rotational response.

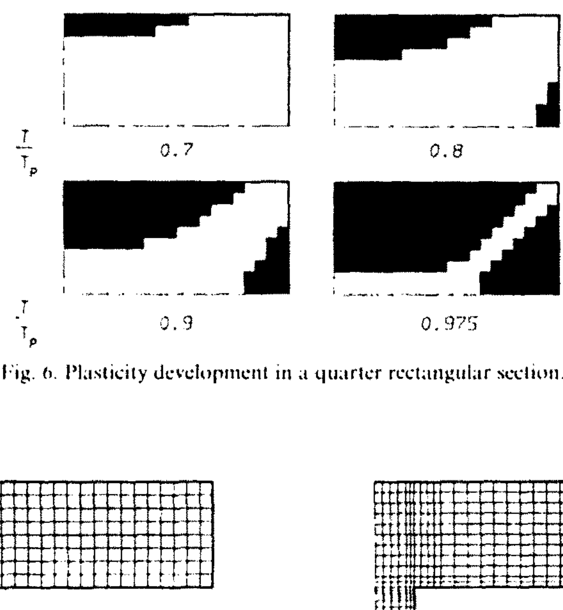

Fig. 6. Plasticity development in a quarter rectangular section.

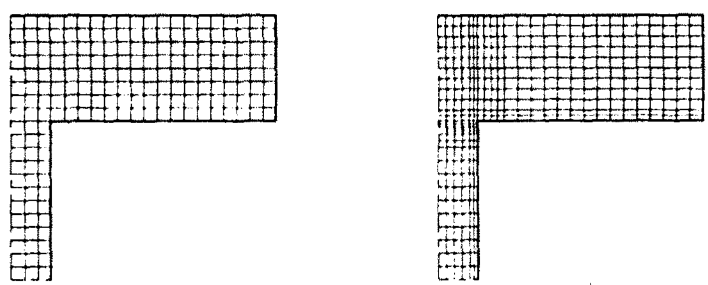

Fig. 7. Quarter I-section cell divisions.

case, is seen to vary considerably between the two cell analogy analyses which is a reflection of the difficulty experienced by the coarser analysis in accurately modelling the elastic stress concentration at the internal corner. However, the subsequent elastic-plastic response is largely unaffected by the predicted point of the initial yield, as is illustrated by the subsequent correspondence of the two sets of cellular analogy results.

The rotational responses given by the cellular analogy and finite element analyses are generally similar in nature but the finite element analysis predicts both a rather higher fully plastic capacity for the section and a somewhat stiffer elastic-plastic response. Both of these features reflect the lower bound nature of the cellular analogy approach and may be related to the treatment of the shear stress distribution in the region of the internal corner. In this area of the section the resultant shear stresses are not parallel to either of the intersecting edges of the section and assumption (c) of the elastic-plastic cellular analogy analysis is therefore invalid in this region. The application of assumption (c) in the internal corner

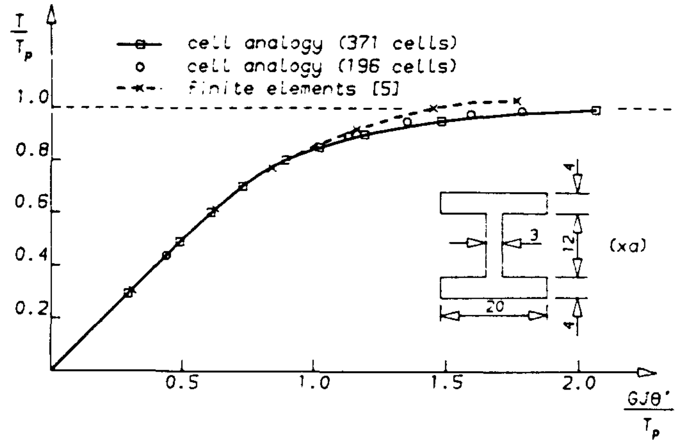

Fig. 8. I-Section elastic–plastic rotational response.

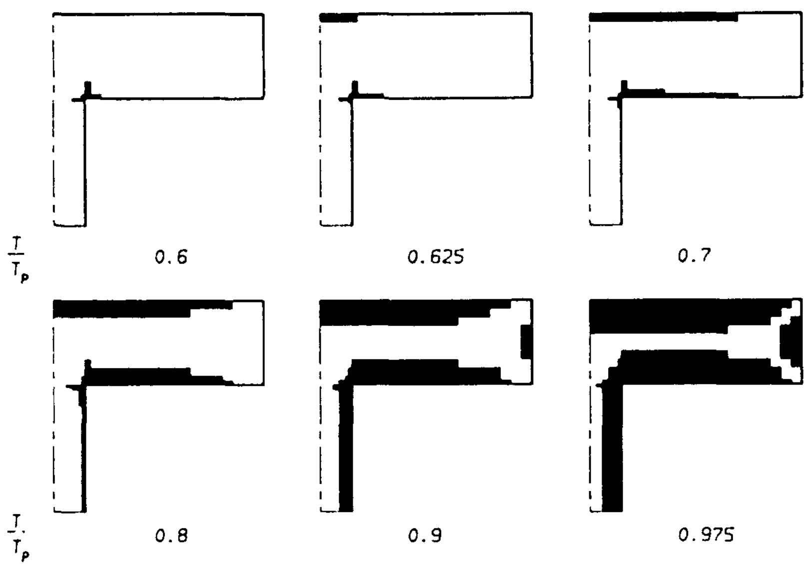

Fig. 9. Plasticity development in a quarter I-section.

vicinity will, in fact, result in the underestimation of the maximum elastic shear stresses in this area. This explains (Fig. 9) the relative reluctance of plasticity to develop inwards from the internal corner, except along continuations of the edges, where the assumption of stress direction being parallel to an edge will be approximately fulfilled.

The elastic shear stresses along directions emanating from an external corner, say along the diagonals of a rectangle, will also not be parallel to a section edge. However, in this case such stresses are not critical and plasticity does not grow from the corner but rather approaches from either side (Fig. 6). The elastic stresses at the interface of the encroaching plastic regions will be edge-parallel and the type of error experienced at internal corners does not therefore arise.

COMPUTATIONAL ASPECTS

The cellular analogy method is extremely straightforward and efficient from a computational point of view. The effort involved in determining the equivalent cell properties and in the determination of the shear stresses from the calculated shear flows is trivial. The linear equations to be solved (eqn (3)) will generally be similar in number to those required

Table 3. Execution time comparison

Method	No. of variables	Time (s)	Time variable (s)
Cellular analogy	100	3.28	0.033
Linear FE—warping function	108	7.11	0.066
Quadratic FE—warping function	56	7.38	0.132

for a finite element solution based on a linear warping displacement function. As with the finite element solution, the linear equation set will be banded and symmetric and similar bandwidths will generally be obtained for given section sub-divisions. However, the introduction of additional cells will not require node renumbering with the cell analogy approach, since the bandwidth is only dependent upon the cell ordering and the nodal numbering is therefore immaterial.

An indication of computational efficiency of the cellular analogy may be obtained from Table 3. The table presents execution times for elastic analyses of the L-section (Fig. 4) which resulted in solutions of comparable accuracy (Table 2). The time/variable may be seen to increase with the complexity of the representation. However, the reduced number of variables required by the quadratic finite element method will make this the preferred approach for more complex problems, for which the simpler formulations would require a very large number of variables to achieve an acceptable degree of accuracy.

A particular attraction of the cell analogy approach is that the solution becomes progressively quicker as plasticity is incorporated. Following each solution, a further cell enters the plastic zone and its shear flow thereafter remains constant. On subsequent re-analysis, the number of unknown shear flows in eqn (3) is therefore reduced and the solution efficiency is thereby improved.

CONCLUSIONS

A cellular analogy has been presented, which has been shown to result in a particularly simple formulation of the elastic–plastic Saint-Venant torsion problem. The rectangular shape adopted for the analogous cell limits the method to the treatment of rectilinearly bounded sections and the assumption of plastic shear stress orientations being parallel to a boundary has been shown to result in the non-rigorous treatment of sections with re-entrant corners.

In common with other stress based methods, the cellular analogy approach does not readily yield information on warping displacements and is thus not readily extended to non-uniform torsional analysis. The method can be applied to multiply-connected sections by including the cut-outs of such sections as additional cells, one cell for each cut-out portion.

Despite its limitations, the ease of application of the cellular method should commend itself and the analogy provides a perhaps unusual illustration of an application of thin-walled structural theory to the analysis of solids.

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